Hypergeometric Functions for Statistical Computing

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Abstract

We perform familiar statistical computations transparently with hypergeometric functions. The program was suggested by Shaw, we offer some commentary.

1 Introduction

The hypergeometric function is given by

\[ _pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \]

where \((a)_n = a(a + 1) \cdots (a + n - 1), (a)_0 = 1.\)

It is implemented in J as the hypergeometric conjunction \(H.\),

\[(a_1 \ldots a_p \ H. \ b_1 \ldots b_q) \ z = _pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z).\]

2 Statistics

2.1 Erf

We have [9]:

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2} dt \\
= \frac{2z}{\sqrt{\pi}} _1F_1 \left( \frac{1}{2}; \frac{3}{2}; -z^2 \right) \\
= \frac{2ze^{-z^2}}{\sqrt{\pi}} _1F_1 \left( 1; \frac{3}{2}; z^2 \right)
\]


The former has convergence problems, as pointed out by Shaw [7]. We can see this experimentally, viz.
erfbad =: {{ 2p_0.5 * y * (0.5 H. 1.5) (-y^2) }}
erfbad 7
_814.656
The latter:
erf =: {{ 2p_0.5 * y * (^(-y^2)) * (1 H. 1.5) (y^2) }}

### 2.2 Normal CDF

The above allows us to define the cdf for the standard normal \(N(0,1)\)\(^3\), viz.
cdf =: {{ -: >: erf (y%%:2) }}

This is equivalent to the approach was used by Marsaglia in 2004\(^5\), independently of Shaw\(^7\).

### 2.3 t-Distribution CDF

For \(\nu\) degrees of freedom we have\(^1\):
\[
\frac{1}{2} + x \frac{\Gamma\left(\frac{1}{2}(\nu + 1)\right)}{\sqrt{\pi \nu} \Gamma\left(\frac{\nu}{2}\right)} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}(\nu + 1); 3; -\frac{x^2}{\nu}\right)
\]
for \(|x| < \sqrt{\nu} \) \( {}_2F_1 \) converges if and only if \(|z| < 1 \).
Hui\(^4\) gives the J dyad:
cdf =: 4 : 0
   assert. (%:x)>|y
   0.5 + y * (!-:x-1) * ((0.5,-:1+x) H. 1.5 (-*:y)%x) % (%:o.x) * !<:-:x
   (note that !n is \(\Gamma(n + 1)\))

### 2.4 \(\chi^2\)-distribution

The CDF for \(r\) degrees of freedom is given by\(^8\)
\[
\frac{\gamma\left(\frac{r}{2}, \frac{x}{2}\right)}{\Gamma\left(\frac{r}{2}\right)}
\]
where \(\gamma\) is the incomplete gamma function\(^10\)
\[
\gamma(a, x) = \int_0^x t^{a-1}e^{-t}dt
= a^{-1}x^ae^{-x} {}_1F_1(a; 1 + a; -x)
\]
We have the following elegant J implementation\(^2\)
gamma =: !@:<:;
incgam =: {{ (1 H. (1+x) % x&((* ^) * (^-)~)) y }} % gamma@[
chisqcdf =: incgam&-:

2
3 Conclusion

Implementations in J are fairly transparent; we do not need to translate mathematical notions to loops.

References


