Hypergeometric Functions for Statistical Computing

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Abstract

We perform familiar statistical computations transparently with hypergeometric functions. The program was suggested by Shaw, we offer some commentary.

1 Introduction

The hypergeometric function is given by

\[ pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \]

where \((a)_n = a(a + 1) \cdots (a + n - 1), (a)_0 = 1\).

It is implemented in J as the hypergeometric conjunction \(H\),

\[ H(a_1 \ldots a_p; b_1 \ldots b_q; z) = pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z). \]

2 Statistics

2.1 Erf

We have [9]:

\[ \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt \]

\[ = \frac{2z}{\sqrt{\pi}} \, _1F_1 \left( \frac{1}{2}; \frac{3}{2}; -z^2 \right) \]

\[ = \frac{2z e^{-z^2}}{\sqrt{\pi}} \, _1F_1 \left( 1; \frac{3}{2}; z^2 \right) \]


The former has convergence problems, as pointed out by Shaw [7]. We can see this experimentally, viz.
erfbad =: {{ 2p_0.5 * y * (0.5 H. 1.5) (-y^2) }}
erfbad 7
814.656

The latter:
erf =: {{ 2p_0.5 * y * (^(-y^2)) * (1 H. 1.5) (y^2) }}

### 2.2 Normal CDF

The above allows us to define the cdf for the standard normal \(N(0,1)\) [3], viz.
cdf =: {{ -: >: erf (y%:2) }}

This is equivalent to the approach was used by Marsaglia in 2004 [5], independently of Shaw [7].

### 2.3 t-Distribution CDF

For \(\nu\) degrees of freedom we have [1]:

\[
\frac{1}{2} + x \frac{\Gamma\left(\frac{1}{2}(\nu+1)\right)}{\sqrt{\pi}\nu\Gamma\left(\frac{\nu}{2}\right)} _2F_1\left(\frac{1}{2}, \frac{1}{2}(\nu+1); \frac{3}{2}; -\frac{x^2}{\nu}\right)
\]

for \(|x| < \sqrt{\nu} \ _2F_1 \text{ converges if and only if } |z| < 1\).

Hui [4] gives the J dyad:
cdf =: 4 : 0
assert. (%:x)>|y
0.5 + y * (!:-:x-1) * ((0.5,-:1+x) H. 1.5 (-*:y)%x) % (%:o.x) * !<:-:x
(note that !n is \(\Gamma(n+1)\))

### 2.4 \(\chi^2\)-distribution

The CDF for \(r\) degrees of freedom is given by [8]

\[
\frac{\gamma\left(\frac{r}{2}, \frac{x^2}{2}\right)}{\Gamma\left(\frac{r}{2}\right)}
\]

where \(\gamma\) is the incomplete gamma function [10]

\[
\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt = a^{-1}x^ae^{-x} _1F_1(a; 1 + a; -x)
\]

We have the following elegant J implementation [2]
gamma =: !@:<:
ingam =: {{ (1 H. (1+x) % x&((* ~) * (~-~)) y } } % gamma@[
chisqcdf =: incgam&-:
3 Conclusion

Implementations in J are fairly transparent; we do not need to translate mathematical notions to loops.

References


