Course-of-Value Recursion via Laziness

V. E. McHale

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Euler gives a recurrence relation for the Catalan numbers:

$$C_n = \sum_{i=1}^{n} C_{i-1} C_{n-i}$$

This uses course-of-value recursion (corresponds to strong induction). Turner motivated laziness by being able to better deal with infinite data structures [2], and in fact we can compute the above sequence all in one go with an infinite return value:

```
module Comb where

catalan :: [Integer]
catalan = 1 : 1 : [sum [(−1) ↑ (k + 1) ∗ (pc (n − ((k ∗ (3 ∗ k − 1)) /. 2)) + pc (n − ((k ∗ (3 ∗ k + 1)) /. 2)))
  where
    pc m | m ⩾ 0 = catalan !! m | otherwise = 0
    infixl 6 /.
    (/.) = quot
```

If it is not obvious this is efficient (and why simply calling the function would not be), the reader can mull over the classic Fibonacci example:

```
fibs :: [Integer]
fibs = 1 : 1 : zipWith (+) fibs (tail fibs)
```

Reinhard Zumkeller [1] uses such a technique to count the number of unlabeled rooted trees with n nodes, for instance.

Tables and functions are slightly mismatched, though they correspond exactly: the values in the table are filled by the function-as-procedure, but infinite tables are available only to God. Laziness allows us to define unbounded tables, bringing functions closer to the set theoretic definition where one identifies a function with its graph. This facility for course-of-value recursion is elegant and special to laziness. That Python (for instance) supports memoization to deal with such problems attests to their importance. Memoization may look ad hoc and laziness is more abstract; this example shows the cosmic place of the former and helps us grasp the latter.

References

[1] OEIS Foundation Inc. Number of unlabeled rooted trees with n nodes, entry a000081 in the on-line encyclopedia of integer sequences. `https://oeis.org/A000081`, 2023.

[2] D. A. Turner. Miranda: A non-strict functional language with polymorphic types. In Jean-Pierre Jouannaud, editor, *Functional Programming Languages and Computer Architecture*, pages 1–16, Berlin, Heidelberg, 1985. Springer Berlin Heidelberg.