Noether's Theorem for Functionals Depending on Higher-Order Derivatives

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August 30, 2016

Abstract

Presents proofs of some theorems involved in the calculus of variations with fields depending on higher order derivatives. We follow Gelfand and Fomin very closely.

1 The Euler-Lagrange Equations in general

Suppose we have a functional of the form $J[u] = \int F(x_i; u_j; \frac{\partial u_j}{\partial x_i}; \frac{\partial^2 u_j}{\partial x_j x_k}; \cdots) dx_1 \cdots dx_n$. We wish to find a sufficient condition that it is stationary.

2 Calculation of $\delta u_{x_i x_j}$

We get that

$$(\bar{\delta u})_{x_i x_j} + \sum_{j,k=1}^{n} u_{x_i x_j x_k} \delta x_k$$

3 General Expression for the variation of a functional

4 Conserved flows

Let us suppose we are given a functional $J[u]$ which is invariant under a transformation of the form

$$x_i^* = \Phi_i(x, u, \partial_i u, \partial_i \partial_j u; \epsilon)\ x_i + \epsilon \phi_i(x, u, \partial_i u)$$
$$u_j^* = \Psi_i(x, u, \partial_i u, \partial_i \partial_j u; \epsilon)\ x_i + \epsilon \psi_j(x, u, \partial_i u)$$

that is, $\int F^*(u*) dx^* = \int F(u) dc$.

Then

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} M = 0$$

whenever u_j are chosen to be extremal.