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Symplectic Geometry & Quantization

1. Symplectic manifold: manifold w/ a closed, nondegenerate 2-form ω:
   i) $d\omega = 0$
   ii) if $\omega_m(X, Y) = 0 \ \forall Y \in T_m M$, then $X = 0$.

With an energy function H, this defines dynamics on a symplectic manifold

canonical form (on T^*Q: $\omega = \sum dp \wedge dq$)

(Also func symmetries = symplectomorphisms)

Darboux's theorem - all symplectic manifolds of the same dimension are locally the same. (symplectomorphic). to T^*Q, cotangent bundle of the configuration space Q. canonical coordinates - just like mechanics.

Interior product: $i(X)\omega = \omega(X, \cdot)$

2. Hamiltonian Vector Fields

$$i(X_H)\omega = dH$$

H is the energy function, e.g. $H = \frac{1}{2m}p^2 + V(q)$

Together, H and ω determine the dynamics.

3. Poisson Brackets

Fundamental exact sequence:

$$0 \longrightarrow \mathbb{R} \xrightarrow{i} C^\infty(M) \xrightarrow{-j} Ham(M) \longrightarrow 0$$

where $Ham(M)$ is the set of Hamiltonian vector fields on M,

$$j: H \longmapsto X_H = \sum \frac{\partial H}{\partial p_j} \cdot \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \cdot \frac{\partial}{\partial p_j}$$

This is exact b/c if $X_H = 0$, $i(X_H)\omega = 0$ and so $dH = 0$.
But $dH = 0 \Rightarrow H$ const. so $\ker j = \operatorname{Im} i$.

$$\{f, g\} := -i(X_f)\,i(X_g)\,\omega = \omega(X_f, X_g)$$

for $\omega = \sum_i dp_i \wedge dq_i$ this is:

$$\{f, g\} = \sum_j \frac{\partial f}{\partial q_j} \cdot \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial q_j} \cdot \frac{\partial g}{\partial p_j}$$

(note: $\dot{f} = \{f, H\}$ which defines **dynamics** on M. Also, $\dot{H} = \{H, H\} = 0$ so $H = $ const.)

<u>Also note:</u> by Darboux's theorem, this is the only version we care about.

4. Quantization & example

	CM	QM
states	points on a symplectic manifold	vectors in $\mathcal{H}$ (a Hilbert space) → a vector space w/ an inner product & Cauchy completeness
observables	function in $C^\infty(M)$	linear operators on $\mathcal{H}$ (unitary)

Want: given $f \in C^\infty(M)$, find $\hat{f} \in Aut(\mathcal{H})$ s.t.
$\{f_1, f_2\} = i[\hat{f}_1, \hat{f}_2] = i(\hat{f}_1\hat{f}_2 - \hat{f}_2\hat{f}_1)$ & $\hat{f}$ is unitary (i.e. Lie bracket preserved)

Consider $\mathcal{F} \subseteq C^\infty(M)$ a Poisson subalgebra.

Then $\sigma : f \mapsto X_f$ where the X_f have a Lie bracket b/c

$\mathcal{F} \longrightarrow \mathfrak{g} = \mathrm{Lie}\, G$ $\quad [X,Y] = L_X Y$ is a Lie bracket.

Consider $\pi : G \to \mathrm{Aut}\, \mathcal{H}$ a unitary representation.

Then there is a $d\pi : \mathfrak{g} \to \mathrm{Aut}\, \mathcal{H}$

$$f \xmapsto{\sigma} X_f \xmapsto{d\pi} d\pi(X_f) \xmapsto{\pm i} \pm i\, d\pi(X_f)$$

Say $\hat{f} := \pm i\, d\pi(X_f)$

This satisfies

$$\widehat{\{f,g\}} = \mp i [\hat{f}, \hat{g}], \quad \text{(note: } [X_f, X_g] = -X_{\{f,g\}} \text{)}$$

Also: $\dot{\hat{f}} = i[\hat{H}, \hat{f}]$, like the cm version.

Example:

Let $\mathcal{F} = \langle qp, p^2, q^2 \rangle$

Then $\{q^2, p^2\} = 4qp$

$\{qp, p^2\} = 2p^2$

$\{qp, q^2\} = -2q^2$

Set $H_1 = qp$, $H_2 = \frac{1}{2}p^2$, $H_3 = \frac{1}{2}q^2$. Then:

$$X_{H_1} = q\frac{\partial}{\partial q} - p\frac{\partial}{\partial p} \qquad X_{H_2} = p\frac{\partial}{\partial q} \qquad X_{H_3} = -q\frac{\partial}{\partial p}$$

note that:

$$\mathcal{F} \simeq \langle X_{H_1}, X_{H_2}, X_{H_3} \rangle \simeq \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle \simeq sl_2(\mathbb{R})$$

Say $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and we get

$[F, G] = H \qquad [H, F] = 2F \qquad [H, G] = -2G$

like we expect

(skip step: determining π for $SL_2(\mathbb{R})$ Instead compute $d\pi$)

et $\mathcal{H}$ be a Hilbert space w/ basis $\{v_j \mid j \in \mathbb{N}_0\}$

Then, setting

$$H v_j = (j + \tfrac{1}{2}) v_j$$

$$F v_j = -\frac{1}{2\mu} v_{j+2}$$

$$G v_j = \frac{\mu}{2} j(j-1) v_{j-2}$$

we get a representation $d\pi : sl_2(\mathbb{R}) \to \mathrm{Aut}\,\mathcal{H}$ of $sl_2(\mathbb{R})$. In fact, it is $d\pi_W$ where π_W is the weil representation.